Solution 1. We will show by mathematical induction that there is always only one such pairing. For $n = 1$ it is obvious.

Between numbers $2n+1, \dot{;}4n-1$ there is exactly one number that can be written in the form $2^k + 1$. It follows that the number $2n$ must be paired with $2^k + 1 - 2n$. Analogously $2n - 1$ must be paired with $2^k + 2 - 2n$, $2n - 2$ with $2^k + 3 - 2n$ and so on up to the number $2^{k-1} + 1$ which must be paired with 2^k.

If we take away these pairs, we are left with the numbers $1, 2, \ldots, 2^k - 2n$. By induction hypothesis, ther is only one way how to pair these numbers, so we are done.

Solution 2. Let R be the circumradius of the dodecagon. Since the central angle of a regular dodecagon is $30°$, the relevant chord lengths are

$$AC = AK = R, \qquad AE = R\sqrt{3}, \qquad KE = 2R.$$

Indeed, AC and AK subtend central angles of $60°$, AE subtends $120°$, and KE is a diameter.

Moreover, AH bisects angle EAK, because arcs EH and HK have the same length. Thus in triangle AEK, point P is the point where the internal angle bisector from A meets side KE. By the angle bisector theorem,

$$\frac{KP}{KE} = \frac{AK}{AK + AE} = \frac{R}{R + R\sqrt{3}}.$$

Therefore

$$KP = \frac{2R}{1 + \sqrt{3}} = R(\sqrt{3} - 1).$$

Consequently

$$AC + KP = R + R(\sqrt{3} - 1) = R\sqrt{3} = AE,$$

as required.

Solution 2. Reflect point K in line AH, and call the image Q. Since AH bisects angle EAK, point Q lies on segment AE. Also

$$AQ = AK = AC \quad \text{and} \quad QP = KP.$$

It is enough to prove $QP = QE$.

Now $\angle KAE = 90°$ and $AK = AQ$, so triangle AKQ is an isosceles right triangle. Hence $\angle AKQ = 45°$. On the other hand, in triangle AKE we have $\angle AKE = 60°$ and $\angle AEK = 30°$. Thus

$$\angle QKP = 60° - 45° = 15°.$$

Since $QP = KP$, triangle QKP is isosceles, so also $\angle PQK = 15°$, and therefore

$$\angle QPE = 30°.$$

But $\angle QEP = \angle AEK = 30°$, so triangle QPE is isosceles and $QP = QE$. Hence

$$AC + KP = AQ + QP = AQ + QE = AE.$$

Solution 3. If we take, for instance, the equality of the first two fractions, then after cross-multiplying we get $b(bc+1) - a(ca+1) = 0$, which simplifies to $(b-a)(cb+ca+1) = 0$. We get similar equalities for the other pairs. So let us set $a = b = t$, and require $ab + ac = -1$, which gives $c = -(t^2 + 1)/t$. The fractions then all equal $-t$. The substitution is valid for $t \neq 0$. Thus, we have shown the fractions can attain any nonzero value.

If all the fractions were equal to 0, we would have $bc = -1$, $ca = -1$, $ab = -1$, and multiplying these gives $(abc)^2 = -1$, which is not possible.

The answer is therefore every nonzero real number.

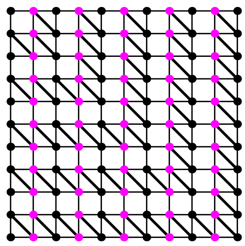
Solution 4. Put coordinates (x, y) on the grid points, where $0 \leq x, y \leq 100$. Call a grid point pink if its x-coordinate is odd. There are

$$50 \cdot 101 = 5050$$

pink points.

Every diagonal of a unit square has endpoints whose x-coordinates differ by 1. Hence every drawn diagonal has exactly one pink endpoint. Since no two drawn diagonals may share an endpoint, the number of drawn diagonals is at most the number of pink points, namely at most 5050.

It remains to show that this bound is attainable — it is met by the example analogous to the one depicted below.



Solution 5. Denote by a, b, c the sides and angles of ABC in the usual way, and by d_1, d_2 the lengths BD, CD, respectively. Clearly $d_1 + d_2 = a$. The angle bisector theorem gives us

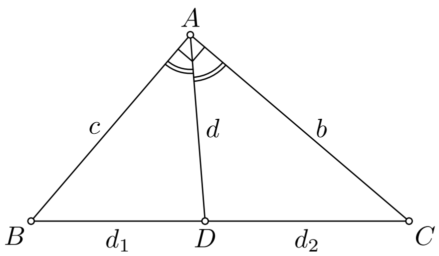
$$\frac{d_1}{d_2} = \frac{c}{b}.$$

By solving this system, we can easily find

$$d_1 = \frac{ab}{b+c}.$$

According to the sine law in ABD, we have

$$\frac{d}{\sin\beta} = \frac{d_1}{\sin 45^\circ}.$$



If we use $\sin\beta = \frac{c}{a}$ and the derived formula for d_1, then after dividing by a we get the equality

$$d(b+c)\sqrt{2} = 2bc. \tag{1}$$

For $b = c = \sqrt{2}$ we have $d = 1$ and clearly $a = 2$, therefore one side of ABC can be rational. We will show that two of them cannot be rational.

If b and c were rational, then from (1), we would have that d would be irrational, which is a contradiction. Now, assume that a is rational and also one of b, c, without loss of generality b. Then $c^2 = a^2 - b^2$ is a rational number. By squaring (1) we can easily get

$$(b+c)^2 = \frac{2b^2c^2}{d^2},$$

therefore even $(b+c)^2$ is rational. But $(b+c)^2 = b^2 + c^2 + 2bc$, therefore even $2bc$ is rational, so both b, c have to be rational, which is a contradiction.

Another solution: We will show a different way of obtaining some relatins between lenghts.

Let E, F be points on the sides AB, AC such that $AEFD$ is a square. For simplicity let $AD = 2$. Then $AE = AF = \sqrt{2}$.

We have $CE = b - \sqrt{2}$. Right triangles CAB and CED are similar with the ratio being $b : (b - \sqrt{2})b$ Therefore we obtain $c = AB = \sqrt{2} \cdot b/(b - \sqrt{2})$ and

$$a^2 = b^2 + c^2 = b^2 + 2b^2/(b - \sqrt(2)^2).$$

From these relations, it easy to observe that there can be at most one rational value between a, b, c.